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A characterization of weighted Besov spaces in quantum calculus

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Abstract

In this paper, subspaces of $L^p(\mathbb{R}_{q,+})$ are defined using q-translations $T_{q,x}$ operator and q-differences operator, called q-Besov spaces. We provide characterization of these spaces by using the q-convolution product.

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1 Introduction

Much recent research activity has focused on the theory and application of Quantum Calculus. This branch of mathematics continues to find new and useful applications. For example the so-called q-analogs of special functions and hypergeometric series, called q-series have many applications in arithmetic theory, combinatorics, quantum physic, group theory [2], and others areas of science and mathematics. Applications of this mathematics include population biology [5], geometric analysis [6], intelligent robotic control [9], approximation theory [22], and financial engineering [21], among others. Our interest in this paper is to characterize some weighted Besov spaces in Quantum Calculus, called weighted q-Besov spaces. In the classical case there are many ways to define Besov spaces see ([4], [20], [24], [25]).

In this paper we express q-Besov spaces in term of convolution $f *_q \varphi_t$ with different kinds of smooth functions φ. These spaces can be described by means of difference differential operator (see [14], [15], [18], [23]).
Throughout the paper weight $w : \mathbb{R}_{q,+} \longrightarrow \mathbb{R}_+$ will be a q-measurable function, $w > 0$ a.e., and we will give a characterization of weighted q-Besov spaces.
Our objective is to find weights where we can get such a characterization of weighted q-Besov spaces

$$\Lambda^{p,m}_{*,w,q} = B^{p,m}_{*,w,\varphi,q} \quad (\text{ with equivalent seminorms })$$

where $\Lambda^{p,m}_{*,w,q}$ the space of even function $f : \mathbb{R}_q \longrightarrow \mathbb{C}$ such that

$$\| f \|_{\Lambda^{p,m}_{*,w,q}} := \left\{ \int_0^{+\infty} \frac{\| \nabla_{q,x} f \|^m_{q,p}}{w(x)^m} \frac{d_q x}{x} \right\}^{\frac{1}{m}} < +\infty, \quad (1 \le m < \infty)$$

and

$$\| f \|_{\Lambda^{p,\infty}_{*,w,q}} := \inf \left\{ C > 0; \| \nabla_{q,x} f \|_{q,p} \le C w(x) \quad a.e.\ x \in \mathbb{R}_{q,+} \right\}, \quad (m = +\infty)$$

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where we have put $\nabla_{q,x} f(y) := T_{q,x} f(y) - f(y)$, and $B^{p,m}_{*,w,\varphi,q}$ the space of the even function $f : \mathbb{R}_q \longrightarrow \mathbb{C}$ belonging to $L^1(\mathbb{R}_{q,+}, \dfrac{d_q x}{(1+x)^2_q})$ satisfying

$$\| f \|_{B^{p,m}_{*,w,\varphi,q}} := \left\{ \int_0^{+\infty} \frac{\| \varphi_t *_q f \|^m_{q,p}}{w(t)^m} \frac{d_q t}{t} \right\}^{\frac{1}{m}} < +\infty, \quad (1 \leq m < +\infty)$$

and

$$\| f \|_{B^{p,\infty}_{*,w,\varphi,q}} := \inf \left\{ C > 0; \| \varphi_t *_q f \|_{q,p} \leq C w(x) \quad a.e \quad t \in \mathbb{R}_{q,+} \right\}, \quad (m = +\infty)$$

where $\varphi_t(x) := t^{-1}\varphi(t^{-1}x)$, $t \in \mathbb{R}_{q,+}$and $x \in \mathbb{R}_+$.

The contents of the paper are as follows. In Section 2, we collect some basic definitions and results about q-harmonic analysis. In Section 3, we give condition about weight and prove the connection between the spaces $\Lambda^{p,m}_{*,w,q}$ and $B^{p,m}_{*,w,\varphi,q}$.

2 Preliminaries

In all the sequel, we assume $q \in (0, 1)$ and we adapt the same notations as in [8].

- A q-shifted factorial is defined by

$$(a; q)_0 := 1, \qquad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k); \;\; n = 1, 2, \cdots, \infty,$$

  and more generally

$$(a_1, \cdots, a_r; q)_n := \prod_{k=1}^{r} (a_k; q)_n.$$

- The basic hypergeometric series or q-hypergeometric series are given for r, s integers by

$${}_r\varphi_s(a_1, \cdots, a_r; b_1, \cdots, b_s; q, x) := \sum_{n=0}^{\infty} \frac{[(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} (a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n} \frac{x^n}{(q; q)_n}.$$

- The q-derivative $D_{q,x} f$ of a function f on an open interval is given by

$$D_{q,x} f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0$$

  and the q-derivative at zero [17] is defined by

$$D_{q,x} f(0) := \lim_{n \to +\infty} \frac{f(xq^n) - f(0)}{xq^n},$$

  where the limit exists and independent of x.

- We also denote
$$[x]_q := \frac{1-q^x}{1-q}, \qquad [n]_q! := \frac{(q;q)_n}{(1-q)^n}.$$
- The q-shift operators are
$$(\Lambda_{q,x}f)(x) := f(qx), \qquad (\Lambda_{q,x}^{-1}f)(x) := \Lambda_{q^{-1},x}f(x).$$
- We consider the q-difference operator
$$\Delta_{q,x} := \Lambda_{q,x}^{-1}D_{q,x}^2.$$
- The q-analogue of $(a+b)^n$ is a non commutative term $(a+b)_q^n$ given by
$$(a+b)_q^n := \begin{cases} a^n(-\frac{b}{a};q)_n, & a \neq 0 \\ q^{n(n-1)/2}b^n, & a = 0. \end{cases}$$
It is clear that $(a+b)_q^n$ and $(b+a)_q^n$ are not always the same. For future use, this definition can be generalized in a way similar to its ordinary counterpart by
$$(1+a)_q^\alpha := \frac{(1+a)_q^\infty}{(1+q^\alpha a)_q^\infty}$$
for any number α.

Proposition 2.1. For $n, k \in \mathbb{N}$

(i) $(a-x)_q^n = (-1)^n q^{\frac{n(n-1)}{2}}(x-q^{-n+1}a)_q^n$

(ii) $D_{q,x}^k(x+a)_q^n = [n]_q.[n-1]_q...[n-k+1]_q(x+a)_q^{n-k}$

(iii) $D_{q,x}^k \dfrac{1}{(1-x)_q^n} = \dfrac{[n]_q.[n+1]_q...[n+k-1]_q}{(1-x)_q^{n+k}}$

Proposition 2.2. For any number α, β

(i) $(1+x)_q^\alpha.(1+q^\alpha x)_q^\beta = (1+x)_q^{\alpha+\beta}$

(ii) $D_{q,x}(1+x)_q^\alpha = [\alpha]_q(1+qx)_q^{\alpha-1}$

(iii) For $\alpha > 0$ and $x = a+b$,
$$2\,(1+[a+b])_q^\alpha \leq q^{\alpha(\alpha-1)/2}[(1+a)_q^\alpha + (1+b)_q^\alpha]$$

- $\mathbb{R}_q := \{\pm q^k, k \in \mathbb{Z}\}$; $\quad \mathbb{R}_{q,+} := \{+q^k, k \in \mathbb{Z}\}$; $\quad \widetilde{\mathbb{R}}_{q,+} := \{+q^k, k \in \mathbb{Z}\} \cup \{0\}$.
- The q-Jackson integral [16] from 0 to a (respectively from a to $+\infty$) is defined by
$$\int_0^a f(x)d_qx := (1-q)a\sum_{n=0}^{+\infty} f(aq^n)q^n, \quad \int_a^\infty f(x)d_qx := (1-q)a\sum_{-\infty}^{-1} f(aq^n)q^n.$$

Remark 2.1. Observe that the q-Jackson integral is a Riemann-Stieltjes integral [1] with respect to a step function having infinitely many points of increase at the points q^k, with the jump at the point q^k being q^k. If we call this step function $\Psi_q(t)$ then $d\Psi_q(t) = d_q t$.

Note that for $n \in \mathbb{Z}$ and $a \in \mathbb{R}_{q,+}$, we have

$$\int_0^{\infty} f(x) d_q x = \int_0^a f(x) d_q x + \int_a^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n,$$

and

$$\int_0^a f\left(\frac{1}{x}\right) d_q x = \int_{qa^{-1}}^{\infty} f(x) \frac{d_q x}{x^2}, \qquad \int_s^{\infty} f\left(\frac{1}{x}\right) \frac{d_q x}{x} = \int_0^{qs^{-1}} f(x) \frac{d_q x}{x}. \tag{2.1}$$

Moreover, if $f \geq 0$ then

$$\int_0^{qs} f(x) d_q x \leq \int_0^s f(x) d_q x, \qquad \int_a^{\infty} f(x) d_q x \leq \int_{qa}^{\infty} f(x) d_q x \tag{2.2}$$

- The following definition of the q-cosine [10] is given by

$$\cos(x; q^2) := {}_1\varphi_1(0, q, q^2; (1-q)^2 x^2) = \sum_{n=0}^{\infty} (-1)^n b_n(x; q^2),$$

where, we have put

$$b_n(x; q^2) := b_n(1; q^2) x^{2n} = q^{n(n-1)} \frac{(1-q)^{2n}}{(q; q)_{2n}} x^{2n}.$$

- The q-cosine Fourier transform $\mathcal{F}_q$ and the q-convolution product are defined for suitable functions f, g as follows

$$\mathcal{F}_q(f)(\lambda) := \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})} \int_0^{\infty} f(t) \cos(\lambda t; q^2) d_q t,$$

$$f *_q g(x) := \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})} \int_0^{\infty} T_{q,x} f(y) g(y) d_q y.$$

Here $T_{q,x}$, $x \in \mathbb{R}_{q,+}$ are the q-even translation operators defined by

$$T_{q,y} f(x) := \sum_{n=0}^{\infty} b_n(y; q^2) \Delta_{q,x}^n f(x) \tag{2.3}$$

Remark that when $q \to 1^-$, the q-translation tends to the classical even translation σ_x given by

$$\sigma_x(f)(y) := \frac{1}{2}[f(x+y) + f(x-y)], \quad y \in [0, +\infty).$$

which has the following properties.

Proposition 2.3. For all $f, g \in L^1(\mathbb{R}_{q,+})$:

(i) $T_{q,x}f(y) = T_{q,y}f(x)$

(ii) $\Delta_{q,x}T_{q,x}f(y) = \Delta_{q,y}T_{q,y}f(x)$

(iii) $\displaystyle\int_0^\infty T_{q,x}f(y)d_qy = \int_0^\infty f(y)d_qy$

(iv) $\displaystyle\int_0^\infty T_{q,x}f(y)g(y)d_qy = \int_0^\infty f(y)T_{q,x}g(y)d_qy$

(v) $T_{q,x}\cos(ty;q^2) = \cos(tx;q^2)\cos(ty;q^2), \quad x, y, t \in \mathbb{R}_{q,+}$.

In [7] the q-cosine Fourier transform satisfies the following:

Theorem 2.1. For $f \in L^1(\mathbb{R}_{q,+})$, $\mathcal{F}_q(f) \in \mathcal{C}_{*,q,0}(\mathbb{R}_q)$ and
$\| \mathcal{F}_q(f) \|_{\mathcal{C}_{*,q,0}} \leq \dfrac{1}{(q(1-q))^{\frac{1}{2}}(q;q)_\infty}||f||_{q,1}$.

Theorem 2.2. (Inversion formula)
(i) Let $f \in L^1(\mathbb{R}_{q,+})$ such that $\mathcal{F}_q(f) \in L^1(\mathbb{R}_{q,+})$, then for all $x \in \mathbb{R}_{q,+}$, we have

$$f(x) = \frac{(1+q^{-1})^{\frac{1}{2}}}{\Gamma_{q^2}(\frac{1}{2})}\int_0^\infty \mathcal{F}_q(f)(y)\cos(xy;q^2)d_qy. \tag{2.4}$$

(ii) $\mathcal{F}_q$ is an isomorphism of $\mathcal{S}_{*,q}(\mathbb{R}_q)$ and ${\mathcal{F}_q}^2 = Id$.

They proved that $\mathcal{F}_q$ can be extended to $L^2(\mathbb{R}_{q,+})$ and we have

Theorem 2.3. (q-Plancherel theorem type)
$\mathcal{F}_q$ is an isomorphism of $L^2(\mathbb{R}_{q,+})$, we have $||\mathcal{F}_q(f)||_{q,2} = ||f||_{q,2}$, for $f \in L^2(\mathbb{R}_{q,+})$ and ${\mathcal{F}_q}^{-1} = \mathcal{F}_q$.

In [13], the authors proved that

Proposition 2.4. For $f, g \in L^1(\mathbb{R}_{q,+})$ we have

(i) $\mathcal{F}_q(f *_q g) = \mathcal{F}_q(f)\mathcal{F}_q(g)$

(ii) $\displaystyle\int_0^\infty \mathcal{F}_q(f)(\xi)g(\xi)d_q\xi = \int_0^\infty f(\xi)\mathcal{F}_q(g)(\xi)d_q\xi$

(iii) $\mathcal{F}_q(T_{q,x}f)(\xi) = \cos(\xi x;q^2)\mathcal{F}_q(f)(\xi))$

(iv) For $f \in L^p(\mathbb{R}_{q,+})$, $g \in L^1(\mathbb{R}_{q,+})$ then $f *_q g \in L^p(\mathbb{R}_{q,+})$ and $||f *_q g||_{q,p} \leq ||f||_{q,p}||g||_{q,1}$.

Specially, we choose $q \in [0, q_0]$ where q_0 is the first zero of the function [11]: $q \mapsto {}_1\varphi_1(0, q, q; q)$ under the condition $\frac{\log(1-q)}{\log q} \in \mathbb{Z}$.
Let us now introduce some q-functional spaces which one will need in this work.

- $\mathcal{C}_{*,q,0}(\mathbb{R}_q)$ the space of even functions f defined on $\mathbb{R}_q$ continuous at 0, and satisfying

$$\lim_{x\to\infty} f(x) = 0 \text{ and } \| f \|_{\mathcal{C}_{*,q,0}} = \sup_{x\in\mathbb{R}_q} |f(x)| < +\infty.$$

- $\mathcal{C}^m_{*,q}(\mathbb{R}_q)$ the space of even functions m times q-differentiable on $\mathbb{R}_q$, continuous at 0. We equip this space with the topology of the uniform convergence of the functions and their q-derivatives.

- $\mathcal{S}_{*,q}(\mathbb{R}_q)$ the q-analogue of Schwartz space formed by the functions $f \in \mathcal{C}^\infty_{*,q,0}(\mathbb{R}_q)$ such that

$$\forall k, n \in \mathbb{N}, \ \ N_{q,n,k}(f) = \sup_{x\in\mathbb{R}_q} (1 + x^2)^n \mid D^k_{q,x} f(x) \mid < +\infty.$$

- $L^p(\mathbb{R}_{q,+}),\ p \in [1, +\infty]$, the space of functions f such that $\| f \|_{p,q} < +\infty$, where

$$\| f \|_{p,q} = (\int_0^\infty |f(x)|^p d_q x)^{\frac{1}{p}} < +\infty, \text{ for } p < \infty,$$

  and

$$\| f \|_{q,\infty} = ess \sup_{x\in\mathbb{R}_{q,+}} \mid f(x) \mid < +\infty.$$

- $\mathcal{S}_{*,q,0}$ the space of even functions $f \in \mathcal{S}_{*,q}$ such that $\displaystyle\int_0^{+\infty} f(x)d_q x = 0$.

- $\mathcal{A}_{*,q}$ the space of even function $\varphi \in \mathcal{S}_{*,q,0}$ such that $\displaystyle\int_0^{+\infty} (\mathcal{F}_q\varphi(t\xi))^2 \frac{d_q t}{t} = 1$ for $\xi \in \mathbb{R}_{q,+}$.

- $\mathcal{A}_{*,1,q}$, the space of even function $\varphi \in \mathcal{A}_{*,q}, supp\varphi \subseteq [0, 1]$, such that $\displaystyle\int_0^{+\infty} x\varphi(x)d_q x = 0$.

To establish the results of the paper, let us first the following notions.

- A weight w is said to satisfy q-Dini condition if there exists $C > 0$ such that

$$\int_0^s \frac{w(t)}{t} d_q t < Cw(s); \quad a.e. \quad s \in \mathbb{R}_{q,+}.$$

- A weight w is said to be a $(b_{1,q})$-weight if there exists $C > 0$ such that

$$\int_{qs}^{+\infty} \frac{w(t)}{t^2} d_q t < C\frac{w(s)}{s}; \quad a.e. \quad s \in \mathbb{R}_{q,+}.$$

We also put $\mathcal{W}_{0,1;q}$ the space of $(b_{1,q})$-weight satisfy q-Dini condition.

Definition 2.1. Let $\varepsilon \geq 0$, $\delta \geq 0$ and w be a weight.
w is said to be a $(d_{\varepsilon;q})$-weight if there exists $C > 0$ such that

$$\int_0^s t^{\varepsilon} w(t) \frac{d_q t}{t} \leq C s^{\varepsilon} w(s); \quad \text{a.e. } s > 0$$

w is said to be $(b_{\delta;q})$ weight if there exist $C > 0$

$$\int_{qs}^{+\infty} \frac{w(t)}{t^{\delta}} \frac{d_q t}{t} \leq C \frac{w(s)}{s^{\delta}}; \quad \text{a.e. } s > 0.$$

We write $\mathcal{W}_{\varepsilon,\delta;q} = (d_{\varepsilon;q}) \cap (b_{\delta;q})$.

Proposition 2.5. Let $\varepsilon \geq 0$, $\delta \geq 0$ and w be a weight, we have

(i) If $w \in (d_{\varepsilon;q})$ then $w \in (d_{\varepsilon';q})$ for any $\varepsilon' > \varepsilon$

(ii) If $w \in (b_{\delta;q})$, then $w \in (b_{\delta';q})$ for any $\delta' > \delta$

(iii) If $\overline{w}(t) = w(t^{-1})$; then $w \in (b_{\varepsilon;\,q})$, if and only if, $\overline{w} \in (d_{\varepsilon;\,q})$

(iv) If $w \in \mathcal{W}_{\varepsilon,\delta;q}$; then $w(t) \geq C \min(t^{-\varepsilon}, t^{\delta})$; $C > 0$.

Proof. The assertions (i) and (ii) follows after minor computations. Let prove (iii),
($\Rightarrow$) Let $w \in (b_{\varepsilon;\,q})$, from (2.1) we have

$$\begin{aligned}
\int_0^s t^{\varepsilon} \overline{w}(t) \frac{d_q t}{t} &= \int_0^s t^{\varepsilon} w(\frac{1}{t}) \frac{d_q t}{t} = \int_{qs^{-1}}^{\infty} \frac{w(u)}{u} \frac{d_q u}{u} \\
&\leq C \frac{w(s^{-1})}{s^{-1}} \\
&\leq C s^{\varepsilon} \overline{w}(s)
\end{aligned}$$

($\Leftarrow$) Let now $\overline{w} \in (d_{\varepsilon;\,q})$, again from (2.1)

$$\begin{aligned}
\int_{qs}^{\infty} \frac{w(t)}{t^{\delta}} \frac{d_q t}{t} &= \int_0^{s^{-1}} \frac{w(\frac{1}{x})}{\frac{1}{x^{\delta}}} \frac{d_q x}{x} = \int_0^{s^{-1}} x^{\delta} \overline{w} x \frac{d_q x}{x} \\
&\leq C s^{-\delta} \overline{w}(s^{-1}) \\
&\leq C \frac{w(s)}{s^{\delta}}
\end{aligned}$$

then $w \in (b_{\varepsilon;\,q})$.
To prove (iv) we use the fact that w is a wight then there exist $s_1, s_2 > 0, C_1, C_2$ such that

$$\begin{aligned}
w(s) &\geq cs^{\delta}. \int_s^{\infty} \frac{w(t)}{t^{\delta}} \frac{d_q t}{t} = cs^{\delta}.A(s) \geq cs^{\delta}.A(s_1) \geq C_1 s^{\delta} \\
w(s) &\geq cs^{-\varepsilon}. \int_0^{\varepsilon} w(t) t^{-\varepsilon} \frac{d_q t}{t} = cs^{-\varepsilon}.B(s) \geq cs^{-\varepsilon}.B(s_2) \geq C_2 s^{-\varepsilon}
\end{aligned}$$

then $w(s) \geq C\min(s^{-\varepsilon}, s^{\delta})$, where $C = \max(C_1, C_2)$. □

Lemma 2.1. Every function $f \in \mathcal{S}_{*,q}$, can be written as

$$f(x) = \int_0^{+\infty} f *_q \varphi_t *_q \varphi_t(x)\frac{d_qt}{t}; \text{ for all } \quad \varphi \in \mathcal{A}_{*,q}.$$

Proof. Let $g(x) = \int_0^{+\infty} f *_q \varphi_t *_q \varphi_t(x)\frac{d_qt}{t}$, then

$$\begin{aligned}
\mathcal{F}_q g(\lambda) &= \int_0^{+\infty}\int_0^{+\infty} f *_q \varphi_t *_q \varphi_t(x)t^{-1}\cos(\lambda x; q^2)d_qxd_qt \\
&= \int_0^{+\infty} \mathcal{F}_q(f *_q \varphi_t *_q \varphi_t)(\lambda)t^{-1}d_qt \\
&= \int_0^{+\infty} \mathcal{F}_q(f)(\lambda)(\mathcal{F}_q(\varphi_t)(\lambda))^2t^{-1}d_qt \\
&= \mathcal{F}_q(f)(\lambda)\int_0^{+\infty}(\mathcal{F}_q(\varphi)(t\lambda))^2\frac{d_qt}{t} = \mathcal{F}_q(f)(\lambda).
\end{aligned}$$

From the fact that $\varphi \in \mathcal{A}_{*,q}$ and the use of the inversion formula we obtain the result. □

Proposition 2.6. Let $\varphi \in \mathcal{A}_{*,q}$ and $\psi \in \mathcal{S}_{*,q}$, then for all $\xi \in \mathbb{R}_{q,+}$

$$\mathcal{F}_q\psi(\xi) = \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t *_q \psi)(\xi)\frac{d_qt}{t}.$$

Indeed, since $\varphi \in \mathcal{A}_{*,q}$, so

$$\begin{aligned}
1 &= \int_0^{+\infty}(\mathcal{F}_q\varphi(t\xi))^2\frac{d_qt}{t} \\
&= \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t)(\xi)\frac{d_qt}{t},
\end{aligned}$$

then from Lemma 2.1, and the first relation in Proposition 2.4, we can see easily that

$$\begin{aligned}
\mathcal{F}_q\psi(\xi) &= \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t)(\xi)\mathcal{F}_q\psi(\xi)\frac{d_qt}{t} \\
&= \int_0^{+\infty} \mathcal{F}_q(\varphi_t *_q \varphi_t *_q \psi)(\xi)\frac{d_qt}{t}.
\end{aligned}$$

Remark 2.2. The last proposition shows that
$\psi_{\varepsilon,\delta;q} = \int_\varepsilon^\delta \varphi_t *_q \varphi_t *_q \psi\frac{d_qt}{t}$ converges to ψ in $\mathcal{S}_{*,q}$ as $\varepsilon \to 0$ and $\delta \to \infty$.

On the same way we recall the q-Calderón's formula studied and given in [19]

Theorem 2.4. Let $f \in L^1(\mathbb{R}_{q,+}, \frac{d_q x}{(x+1)_q^2})$ and $\varphi \in \mathcal{A}_{*,q}$. For $0 < \varepsilon < \delta$ define

$$f_{\varepsilon,\delta;q}(x) = \int_{\varepsilon}^{\delta} (\varphi_t *_q \varphi_t *_q f)(x) \frac{d_q t}{t}.$$

Then $f_{\varepsilon,\delta;q}$ converge to f in $\mathcal{S}'_{q,*,0}$ as $\varepsilon \to 0$ and $\delta \to \infty$.

To finish this preliminary section let us state the chief tool in our investigation, that is the Schur lemma [3] that will be useful for our purposes.

Lemma 2.2. (Schur lemma [3]) Let $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and let $K : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}_+$ be a measurable function and write $T_K(f)$ for

$$T_K(f)(w_2) = \int_{\Omega_1} K(w_1, w_2) f(w_1) d\mu_1(w_1).$$

If there exist $C > 0$ and measurable function $h_i : \Omega_i \longrightarrow \mathbb{R}_+ (i = 1, 2)$ such that

$$\int_{\Omega_1} K(w_1, w_2) h_1^{p'}(w_1) d\mu_1(w_1) \leq C h_2^{p'}(w_2); \quad \mu_2 - a.e, \tag{2.5}$$

$$\int_{\Omega_2} K(w_1, w_2) h_2^{p}(w_2) d\mu_2(w_2) \leq C h_1^{p}(w_1); \quad \mu_1 - a.e. \tag{2.6}$$

Then T_K defines a bounded operator from $L^p(\Omega_1, \mu_1)$ into $L^p(\Omega_2, \mu_2)$.

3 Characterization of q-Besov spaces

We begin first by establish some general technical results that can be used to relate properties about q-difference $\nabla_{q,x} f$ and q-convolutions $\varphi_t *_q f$.

Lemma 3.1. For all $y \in [0, 1]$ and $\varphi \in \mathcal{A}_{*,q}.$, we have

$$\|\nabla_{q,y}\varphi\|_{1,q} \leq y \int_0^{+\infty} |D_{q,y}(T_{q,x}\varphi)(0)| d_q x.$$

Proof. Let $y \in [0, 1]$ and $\varphi \in \mathcal{A}_{*,q}$, from Proposition 2.3 we have

$$\begin{aligned} \|\nabla_{q,y}\varphi\|_{1,q} &= \int_0^{\infty} \mid T_{q,y}\varphi(x) - \varphi(x) \mid d_q x \\ &= \int_0^{\infty} y. \left| \frac{T_{q,y}\varphi(x) - T_{q,0}\varphi(x)}{y} \right| d_q x \\ &= y \int_0^{\infty} \left| \frac{T_{q,x}\varphi(y) - T_{q,x}\varphi(0)}{y} \right| d_q x \\ &\leq y \int_0^{\infty} \mid D_{q,y}(T_{q,x}\varphi)(0) \mid d_q x \end{aligned}$$

which leads to the result. ∎

Proposition 3.1. Let $p \in [1,\infty]$, $\zeta \geq 0$ and $\varphi \in \mathcal{A}_{*,q}$. Then there exist $C_q > 0$ such that if $f \in L^1(\mathbb{R}_{q,+}, \frac{d_qx}{(x+1)_q^2})$, then we have that

$$\| \varphi_t *_q f \|_{p,q} \leq C_q \int_0^\infty \min((\frac{x}{t}), (\frac{t}{x})^\zeta) \| \nabla_{q,x} f \|_{p,q} \frac{d_qx}{x} \tag{3.1}$$

and

$$\| \nabla_{q,x} f \|_{p,q} \leq C_q \int_0^\infty \min((\frac{x}{t}), 1) \| \varphi_t *_q f \|_{p,q} \frac{d_qt}{t}. \tag{3.2}$$

Proof. Since $\int_0^{+\infty} \varphi(x) d_qx = 0$, it follows that

$$\varphi_t *_q f(y) = \int_0^{+\infty} \varphi_t(x) \nabla_{q,x} f(y) d_qx,$$

from q-Minkowski's inequality one get

$$\begin{aligned} \|\varphi_t *_q f\|_{p,q} &= (\int_0^{+\infty} | \int_0^{+\infty} \varphi_t(x) \nabla_{q,x} f(y) d_qx|^p d_qy)^{\frac{1}{p}} \\ &\leq \int_0^{+\infty} |\varphi_t(x)| (\int_0^{+\infty} |\nabla_{q,x} f(y)|^p d_qy)^{\frac{1}{p}} d_qx \\ &\leq \int_0^{+\infty} \frac{x}{t} |\varphi(\frac{x}{t})| \|\nabla_{q,x} f\|_{p,q} \frac{d_qx}{x} \\ &\leq \int_0^1 \frac{x}{t} |\varphi(\frac{x}{t})| \|\nabla_{q,x} f\|_{p,q} \frac{d_qx}{x} + \int_1^{+\infty} \frac{x}{t} |\varphi(\frac{x}{t})| \|\nabla_{q,x} f\|_{p,q} \frac{d_qx}{x}. \end{aligned}$$

Hence, the relation (3.1) follows from the trivial estimates, for $y = x/t$

$$\begin{aligned} y^{1+\zeta} |\varphi(y)| &\leq C_q \quad \text{if} \quad y \in [1,\infty), \\ |\varphi(y)| &\leq C_q \quad \text{if} \quad y \in [0,1]. \end{aligned}$$

To prove (3.2) we recall that for $0 < \varepsilon < \delta$,

$$\nabla_{q,x} f_{\varepsilon,\delta}(y) = \int_\varepsilon^\delta (\nabla_{q,x} \varphi_t) *_q \varphi_t *_q f(y) \frac{d_qt}{t} \tag{3.3}$$

Hence q-Minkowski's inequality and q-Young's inequality give

$$\|\nabla_{q,x} f_{\varepsilon,\delta}\|_{p,q} \leq \int_\varepsilon^\delta \|\nabla_{q,x} \varphi_t\|_{1,q} \|\varphi_t *_q f\|_{p,q} \frac{d_qt}{t}$$

Since $\|\nabla_{q,y}\varphi\|_{1,q} \leq 2\|\varphi\|_{1,q}, \ \ y \geq 1$, from Proposition 2.3, for $a = 1/t$ and Lemma 3.1 one has

$$\begin{aligned}
\|\nabla_{q,x}\varphi_t\|_{1,q} &= \int_0^\infty \mid \nabla_{q,x}\varphi_t(y) \mid d_q y = \int_0^\infty \mid T_{q,x}\varphi_t(y) - \varphi_t(y) \mid d_q y \\
&= \int_0^\infty \frac{1}{t} \mid T_{q,x}\left[\varphi(\frac{y}{t})\right] - \varphi(\frac{y}{t}) \mid d_q y \\
&= \int_0^\infty \frac{1}{t} \mid T_{q,\frac{x}{t}}\left(\varphi(\frac{y}{t})\right) - \varphi(\frac{y}{t}) \mid d_q y \\
&= \int_0^\infty \mid T_{q,\frac{x}{t}}\left(\varphi(u)\right) - \varphi(u) \mid d_q u \\
&= \|\nabla_{q,x/t}\varphi\|_{1,q} \\
&\leq C_q \min(1, \frac{x}{t}).
\end{aligned}$$

Therefore, using the previous estimate relation (3.3), we get

$$\begin{aligned}
\|\nabla_{q,x} f_{\varepsilon,\delta}\|_{p,q} &\leq \int_\varepsilon^\delta \|\nabla_{q,x}\varphi_t\|_{1,q} \|\varphi_t *_q f\|_{p,q} \frac{d_q t}{t} \\
&\leq C_q \int_\varepsilon^\delta \min(1, \frac{x}{t}) \|\varphi_t *_q f\|_{p,q} \frac{d_q t}{t}
\end{aligned}$$

Now using the q-Calderon's formula in Theorem 2.4 we obtain $\nabla_{q,x} f_{\varepsilon,\delta} \underset{\substack{\varepsilon\to 0\\ \delta\to\infty}}{\longrightarrow} \nabla_{q,x} f$ in L^p, and then

$$\begin{aligned}
\|\nabla_{q,x} f\|_{p,q} &\leq \int_0^\infty \|\nabla_{q,x}\varphi_t\|_{1,q} \|\varphi_t *_q f\|_{p,q} \frac{d_q t}{t} \\
&\leq C_q \int_0^\infty \min(1, \frac{x}{t}) \|\varphi_t *_q f\|_{p,q} \frac{d_q t}{t}.
\end{aligned}$$

□

Although for the purposes of this paper only a particular case of the following proposition will be used, we shall state a general version of it that we find interesting in its own right.

Proposition 3.2. Given ε, $\delta \in [0, \infty]$, $p \in (1, \infty)$ and w a weight, let us consider

$$\Theta_{\varepsilon,\delta}(s,t) = \frac{w(s)}{w(t)} \min((\frac{s}{t})^\varepsilon, (\frac{t}{s})^\delta).$$

If $w(s) = \lambda^{\frac{1}{p'}}(s)\mu^{-\frac{1}{p}}(s^{-1})$ for some pair of weights λ, $\mu \in \mathcal{W}_{\varepsilon,\delta;q}$, then there exist $C_q > 0$ and $g : \mathbb{R}_{q,+} \longrightarrow \mathbb{R}_+$ q-measurable such that

$$\int_0^\infty \Theta_{\varepsilon,\delta}(s,t) g^{p'}(s) \frac{d_q s}{s} \leq C_q g^{p'}(t), \tag{3.4}$$

and

$$\int_0^\infty \Theta_{\varepsilon,\delta}(s,t) g^p(t) \frac{d_q t}{t} \leq C_q g^p(s). \tag{3.5}$$

Proof. Let us take $g(t)=\lambda^{1/pp'}(t)\mu^{1/pp'}(t^{-1})$. Then $g^{p'}(s)=\lambda(s)/w(s)$ and $g^{p}(t)=w(t)\mu(t^{-1})$.

Therefore from (2.2),

$$\begin{aligned}\int_0^{+\infty}\Theta_{\varepsilon,\delta}(s,t)g^{p'}(s)\frac{d_qs}{s} &= \frac{1}{w(t)}\int_0^{+\infty}\lambda(s)\min((\frac{s}{t})^{\varepsilon},(\frac{t}{s})^{\delta})\frac{d_qs}{s}\\ &= \frac{1}{t^{\varepsilon}w(t)}\int_0^{t}s^{\varepsilon}\lambda(s)\frac{d_qs}{s}+\frac{t^{\delta}}{w(t)}\int_t^{+\infty}\frac{\lambda(s)}{s^{\delta}}\frac{d_qs}{s}\\ &\leq \frac{1}{t^{\varepsilon}w(t)}\int_0^{t}s^{\varepsilon}\lambda(s)\frac{d_qs}{s}+\frac{t^{\delta}}{w(t)}\int_{qt}^{+\infty}\frac{\lambda(s)}{s^{\delta}}\frac{d_qs}{s}\\ &\leq C_q\frac{\lambda(t)}{w(t)}=C_qg^{p'}(t).\end{aligned}$$

On the other hand, same from (2.2),

$$\begin{aligned}\int_0^{+\infty}\Theta_{\varepsilon,\delta}(s,t)g^{p}(t)\frac{d_qt}{t} &= w(s)\int_0^{+\infty}\mu(t^{-1})\min((\frac{s}{t})^{\varepsilon},(\frac{t}{s})^{\delta})\frac{d_qt}{t}\\ &= \frac{w(s)}{s^{\delta}}\int_0^{s}t^{\delta}\mu(t^{-1})\frac{d_qt}{t}+s^{\varepsilon}w(s)\int_s^{+\infty}\frac{\mu(t^{-1})}{t^{\varepsilon}}\frac{d_qt}{t}\\ &= \frac{w(s)}{s^{\delta}}\int_{s^{-1}}^{\infty}\frac{\mu(t)}{t^{\delta}}\frac{d_qt}{t}+s^{\varepsilon}w(s)\int_0^{s^{-1}}t^{\varepsilon}\mu(t)\frac{d_qt}{t}\\ &\leq \frac{w(s)}{s^{\delta}}\int_{qs^{-1}}^{\infty}\frac{\mu(t)}{t^{\delta}}\frac{d_qt}{t}+s^{\varepsilon}w(s)\int_0^{s^{-1}}t^{\varepsilon}\mu(t)\frac{d_qt}{t}\\ &\leq C_q\mu(s^{-1})w(s)=C_qg^{p}(s).\end{aligned}$$

□

We need the following Lemma that we will use after in several of the remaining proofs.

Lemma 3.2. For all $(x,y)\in(q^{\mathbb{Z}},q^{\mathbb{N}})$, $y\neq q$ we have

$$T_{q,y}\left(\frac{1}{(x+1)_q^2}\right)\leq\frac{1}{1-q}.\frac{1}{(x+1)_q^2}.$$

Proof. Let $x,y\in\mathbb{R}_q$, from (2.3) a simple recurrence on $n\in\mathbb{N}$ leads

$$
\begin{aligned}
T_{q,y}\left(\frac{1}{(x+1)_q^2}\right) &= \sum_{n=0}^{\infty} b_n(y;q^2)\Delta_{q,x}^n\left(\frac{1}{(x+1)_q^2}\right)\\
&= \sum_{n=0}^{\infty} q^{n(n-1)}\frac{(1-q)^{2n}}{(q;q)_{2n}}[-2]_q[-3]_q...[-1-2n]_q\frac{y^{2n}}{(q^nx+1)_q^{2n+2}}\\
&= \sum_{n=0}^{\infty} q^{n(n-1)}q^{2n^2+n}[2n+1]_q\frac{y^{2n}}{(q^nx+1)_q^{2n+2}}\\
&= \sum_{n=0}^{\infty} q^{n(n-1)}q^{2n^2+n}[2n+1]_q q^{-n(2n+2)}\frac{y^{2n}}{(x+q^{-n})_q^{2n+1}}
\end{aligned}
$$

from the fact that for $x \in \mathbb{R}_q$

$$
\begin{aligned}
\frac{1}{(x+q^{-n})_q^{2n+1}} &= \frac{1}{(x+q^{-n})(x+q^{-n+1})...(x+q^{-1})}\cdot\frac{1}{(x+1)_q^2}\cdot\frac{1}{(x+q^2)(x+q^3)...(x+q^{n+1})}\\
&\leq \frac{1}{(x+1)_q^2}\; q^{\frac{n(n+1)}{2}}q^{\frac{-n(n+3)}{2}}\\
&\leq \frac{q^{-n}}{(x+1)_q^2}
\end{aligned}
$$

we can deduce that for $y = q^k, k \neq 1$

$$
\begin{aligned}
T_{q,y}\left(\frac{1}{(x+1)_q^2}\right) &= \sum_{n=0}^{\infty} q^{n(n-1)}q^{-n}[2n+1]_q\frac{y^{2n}}{(x+q^{-n})_q^{2n+1}}\\
&\leq \frac{1}{(1-q)}\sum_{n=0}^{\infty} q^{n(n-1)}q^{-n}\frac{y^{2n}}{(x+1)_q^2}\\
&\leq \frac{1}{(1-q)}\cdot\frac{1}{(x+1)_q^2}\cdot\sum_{n=0}^{\infty} q^{n(n-1)}(q^{-2}y^2)^n\\
&\leq \frac{1}{(1-q)}\cdot\frac{1}{(x+1)_q^2}\cdot\frac{1}{1-q^{-2}y^2}\\
&\leq \frac{1}{(1-q)}\cdot\frac{1}{(x+1)_q^2}.
\end{aligned}
$$

the result follow. □

Proposition 3.3. Let $p \in [1,\infty]$ and let f be a q-measurable function.

If $\| \nabla_{q,x} f \|_{p,q} \in L^1(\mathbb{R}_{q,+}, \frac{d_qx}{(x+1)_q^2})$ then $f \in L^1(\mathbb{R}_{q,+}, \frac{d_qx}{(x+1)_q^2})$.

Proof. Let $\Psi \in L^{p'}(\mathbb{R}_{q,+}, d_qx)$ such that $\Psi > 0$ a.e. Then q-Hölder's inequality and q-Fubini's theorem give

$$\int_0^{+\infty} \Big[\int_0^{+\infty} \frac{|T_{q,x}f(y) - f(y)|}{(x+1)_q^2} d_qx\Big] \Psi(y) d_qy < \infty.$$

Therefore,

$$\int_0^{+\infty} \frac{|T_{q,x}f(y) - f(y)|}{(x+1)_q^2} d_qx < \infty \quad \text{for} \quad a.e. \quad y \in \mathbb{R}_{q,+}.$$

Now we have to prove that $f \in L^1(\mathbb{R}_{q,+}, \frac{d_qx}{(x+1)_q^2})$. From Proposition 2.3 and Lemma 3.2

$$\begin{aligned}
\int_0^{+\infty} \frac{| f(x) |}{(x+1)_q^2} d_qx &\leq \int_0^{+\infty} \frac{| T_{q,y}f(x) - f(x) |}{(x+1)_q^2} d_qx + \int_0^{+\infty} \frac{| T_{q,y}f(x) |}{(x+1)_q^2} d_qx \\
&\leq \int_0^{+\infty} \frac{| T_{q,y}f(x) - f(x) |}{(x+1)_q^2} d_qx + \int_0^{+\infty} f(x) \mid T_{q,y}\left(\frac{1}{(x+1)_q^2}\right) d_qx \\
&\leq \int_0^{+\infty} \frac{| T_{q,y}f(x) - f(x) |}{(x+1)_q^2} d_qx + \frac{1}{1-q}\int_0^{+\infty} \frac{| f(x) |}{(x+1)_q^2} d_qx \\
&\leq \int_0^{+\infty} \frac{| T_{q,x}f(y) - f(y) |}{(x+1)_q^2} d_qx + \int_0^{+\infty} \frac{| f(y) |}{(x+1)_q^2} d_qx + \frac{2-q}{1-q}\int_0^{+\infty} \frac{| f(x) |}{(x+1)_q^2} d_qx.
\end{aligned}$$

Since $x \longmapsto \frac{1}{(x+1)_q^2} \in L^1(\mathbb{R}_{q,+}, d_qx)$, then using (3.6), we obtain

$$\begin{aligned}
\frac{1}{q-1}\int_0^{+\infty} \frac{| f(x) |}{(x+1)_q^2} d_qx &\leq \int_0^{+\infty} \frac{| T_{q,y}f(x) - f(x) |}{(x+1)_q^2} \mid d_qx + \int_0^{+\infty} \frac{| f(y) |}{(x+1)_q^2} d_qx \\
&< \infty \quad \text{for} \quad a.e. \quad y \in \mathbb{R}_{q,+}
\end{aligned}$$

which leads to the result. ☐

▶ Now we start the main result of this paper with the case $m = \infty$ which easily follows from Proposition 3.1.

Theorem 3.1. Let $p \in [1, \infty]$, $\varphi \in \mathcal{A}_{*,q}$ and $w \in \mathcal{W}_{0,1;q}$. Then

$$\Lambda^{p,\infty}_{*,w,q} = B^{p,\infty}_{*,w,\varphi,q} \qquad \text{with equivalent seminorms.}$$

Proof. Let $f \in \Lambda^{p,\infty}_{*,w,q}$, then one has

$$\begin{aligned}
\int_0^{+\infty} \frac{\|\nabla_{q,x}f\|_{p,q}}{(x+1)_q^2} d_qx &\leq C_q \int_0^{\infty} \frac{w(x)}{(x+1)_q^2} d_qx \\
&\leq C_q\Big[\int_0^1 w(x)\frac{d_qx}{x} + \int_1^{\infty} w(x)\frac{d_qx}{x^2}\Big] \\
&< \infty,
\end{aligned}$$

what combined with Proposition 3.3 gives $\displaystyle\int_0^\infty \frac{|f(x)|}{(x+1)_q^2} d_qx < \infty$.
Let us prove that $\|\varphi_t *_q f\|_{p,q} \le C_q w(t)$. From (2.2) and (3.1) for $\zeta = 1$ one has

$$\begin{aligned}
\|\varphi_t *_q f\|_{p,q} &\le C_q\Big[\frac{1}{t}\int_0^t \|\nabla_{q,x} f\|_{p,q} d_qx + t\int_t^\infty \|\nabla_{q,x} f\|_{p,q}\frac{d_qx}{x^2}\Big] \\
&\le C_q\Big[\int_0^t \frac{x}{t} w(x)\frac{d_qx}{x} + t\int_t^\infty w(x)\frac{d_qx}{x^2}\Big] \\
&\le C_q\Big[\int_0^t \frac{x}{t} w(x)\frac{d_qx}{x} + t\int_{qt}^\infty w(x)\frac{d_qx}{x^2}\Big] \\
&\le C_q w(t).
\end{aligned}$$

Conversely if $f \in B^{p,\infty}_{*,w,\varphi,q}$, then from (2.2) and (3.2) one has

$$\begin{aligned}
\|\nabla_{q,x} f\|_{p,q} &\le C_q\Big[\int_0^x \|\varphi_t *_q f\|_{p,q}\frac{d_qt}{t} + x\int_x^\infty \|\varphi_t *_q f\|_{p,q}\frac{d_qt}{t^2}\Big] \\
&\le C_q\Big[\int_0^x \frac{w(t)}{t} d_qt + x\int_x^\infty \frac{w(t)}{t^2} d_qt\Big] \\
&\le C_q\Big[\int_0^x \frac{w(t)}{t} d_qt + x\int_{qx}^\infty \frac{w(t)}{t^2} d_qt\Big] \\
&\le C_q w(x).
\end{aligned}$$

□

▶ We prove now the main Theorem in the case $m = 1$.

Theorem 3.2. Let $p \in [1,\infty]$, $\varphi \in \mathcal{A}_{*,q}$ and $w \in \mathcal{W}_{0,1;q}$ such that $\mu(t) = w^{-1}(t^{-1})$. Then

$$\Lambda^{p,1}_{*,w,q} = B^{p,1}_{*,w,\varphi,q} \qquad \text{with equivalent seminorms.}$$

Proof. Assume $f \in \Lambda^{p,1}_{*,w,q}$. Let us first prove that

$$\int_0^\infty \frac{|f(x)|}{(x+1)_q^2} d_qx < \infty.$$

From Proposition 2.5, we have

$$\frac{1}{xw(x)} \ge C_q\frac{1}{x}\min(1,\frac{1}{x}) \ge C_q\frac{1}{x}\min(x,\frac{1}{x}) \ge \frac{C_q}{(x+1)_q^2}.$$

Hence

$$\int_0^\infty \frac{\|\nabla_{q,x} f(x)\|_{p,q}}{(x+1)_q^2} d_qx \le C_q\int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)}\frac{d_qx}{x} < \infty$$

and we apply Proposition 3.3 again.

We will prove that $\|f\|_{B^{p,1}_{*,w,\varphi,q}} \leq C_q \|f\|_{\Lambda^{p,1}_{*,w,q}}$.

Using (2.2) and (3.1) with $\zeta = 1$ we have

$$\begin{aligned}
\int_0^\infty \frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} d_qt &\leq C_q \int_0^\infty \Big[\int_0^\infty \min\big(\frac{x}{t}, \frac{t}{x}\big) \frac{\|\nabla_{q,x} f\|_{p,q}}{w(t)} \frac{d_qx}{x}\Big] \frac{d_qt}{t} \\
&= C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\int_0^\infty \min\big(\frac{x}{t}, \frac{t}{x}\big) \mu(t^{-1}) \frac{d_qt}{t}\Big] \frac{d_qx}{x} \\
&= C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\int_0^x \frac{t\mu(t^{-1})}{x} \frac{d_qt}{t} + \int_x^\infty \frac{x\mu(t^{-1})}{t} \frac{d_qt}{t}\Big] \frac{d_qx}{x} \\
&\leq C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\frac{1}{x} \int_{qx^{-1}}^\infty \mu(t) \frac{d_qt}{t^2} + \int_0^{qx^{-1}} \mu(t) \frac{d_qt}{t}\Big] \frac{d_qx}{x} \\
&\leq C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \Big[\frac{1}{x} \int_{qx^{-1}}^\infty \mu(t) \frac{d_qt}{t^2} + \int_0^{x^{-1}} \mu(t) \frac{d_qt}{t}\Big] \frac{d_qx}{x} \\
&\leq C_q \int_0^\infty \|\nabla_{q,x} f\|_{p,q} \mu(x^{-1}) \frac{d_qx}{x} \\
&\leq C_q \int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \frac{d_qx}{x}.
\end{aligned}$$

Let $f \in B^{p,1}_{*,w,\varphi,q}$, from (2.2), (3.2) and q-Fubini's theorem we get

$$\begin{aligned}
\int_0^\infty \frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \frac{d_qx}{x} &\leq C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^\infty \mu(x^{-1}) \min\big(1, \frac{x}{t}\big) \frac{d_qx}{x}\Big] \frac{d_qt}{t} \\
&= C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^\infty \mu(s) \min\big(1, \frac{1}{st}\big) \frac{d_qs}{s}\Big] \frac{d_qt}{t} \\
&= C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^{t^{-1}} \frac{\mu(s)}{s} d_qs + \int_{t^{-1}}^\infty \frac{\mu(s)}{s^2} d_qs\Big] \frac{d_qt}{t} \\
&\leq C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \Big[\int_0^{t^{-1}} \frac{\mu(s)}{s} d_qs + \int_{qt^{-1}}^\infty \frac{\mu(s)}{s^2} d_qs\Big] \frac{d_qt}{t} \\
&\leq C_q \int_0^\infty \|\varphi_t *_q f\|_{p,q} \mu(t^{-1}) \frac{d_qt}{t} \\
&= C_q \int_0^\infty \frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} \frac{d_qt}{t}.
\end{aligned}$$

□

Theorem 3.3. Let $p \in [1, \infty]$, $m \in (1, \infty)$, and w be a weight such that $w(t) = \lambda^{\frac{1}{m'}}(t)\mu^{-\frac{1}{m}}(t^{-1})$ for some pair of weights λ, $\mu \in \mathcal{W}_{0,1;q}$. Then for $\varphi \in \mathcal{A}_{*,q}$,

$$\Lambda^{p,m}_{*,w,q} = B^{p,m}_{*,w,\varphi,q} \qquad \text{with equivalent seminorms.}$$

Proof. Assume that $f \in \Lambda^{p,m}_{*,w,q}$. We will first show that

$$\int_0^\infty \frac{|f(x)|}{(x+1)_q^2} d_qx < \infty.$$

We denote

$$\Phi(x) = \frac{x}{(x+1)_q^2} w(x)$$

under the assumptions $\lambda, \mu \in \mathcal{W}_{0,1;q}$ one has $\Phi \in L^{m'}(\mathbb{R}_{q,+}, \frac{d_qx}{x})$. Indeed,

$$\int_0^\infty \Phi^{m'}(t)\frac{d_qt}{t} \leq \int_0^\infty \lambda(t)\mu^{-\frac{m'}{m}}(t^{-1})\frac{t^{m'}}{(t+1)_q^{2m'}}\frac{d_qt}{t}.$$

Using Proposition 2.5 we have $\mu(s) \geq C_q \min(1,s)$. Therefore

$$\begin{aligned}\int_0^\infty \Phi^{m'}(t)\frac{d_qt}{t} &\leq \int_0^\infty \lambda(t)\max(1,t^{m'-1})\frac{t^{m'}}{(t+1)_q^{2m'}}\frac{d_qt}{t}\\ &\leq C_q\Big[\int_0^1 \lambda(t)\frac{d_qt}{t} + \int_1^\infty \frac{\lambda(t)}{t}\frac{d_qt}{t}\Big]\\ &< \infty.\end{aligned}$$

Then using q-Hölder's inequality one has

$$\int_0^\infty \frac{\|\nabla_{q,x}f\|_{p,q}}{(x+1)_q^2} d_qx = \int_0^\infty \frac{\|\nabla_{q,x}f\|_{p,q}}{w(x)}\Phi(x)\frac{d_qx}{x} < \infty$$

and we apply Proposition 3.3.

Now we prove that

$$\|f\|_{B^{p,m}_{*,w,\varphi,q}} \leq C_q \|f\|_{\Lambda^{p,m}_{*,w,q}}.$$

From (3.1) in Proposition 3.1 and taking $\rho = 1$, it follows that

$$\frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} \leq \int_0^\infty K(x,t)\frac{\|\nabla_{q,x}f\|_{p,q}}{w(x)}\frac{d_qx}{x}$$

where

$$K(x,t) = \frac{w(x)}{w(t)}\min(1,\frac{t}{x}).$$

If we take

$$(\Omega_1,\mu_1) = (\mathbb{R}_{q,+}, \frac{d_qx}{x})$$

and

$$(\Omega_2, \mu_2) = (\mathbb{R}_{q,+}, \frac{d_q x}{x}).$$

Since $K(x,t) = \Theta_{0,1}(x,t)$, we can apply Proposition 3.2 with $\varepsilon = 0$ and $\delta = 1$ to get a q-measurable function g satisfying (3.4) and (3.5).

Now write $h_1(x) = g(x)$ and $h_2(t) = g(t)$. Obviously, using (3.4) and (3.5) give (2.5) and (2.6) in Lemma 2.2, what shows T_K is bounded from $L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})$ into $L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})$. Therefore,

$$\begin{aligned}
\|f\|_{B^{p,m}_{*,w,\varphi,q}} &\leq C_q \|T_K(\frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)})\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})} \\
&\leq C_q \|\frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)}\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})} \\
&\leq C_q \|f\|_{\Lambda^{p,m}_{*,w,q}}.
\end{aligned}$$

Conversely, let $f \in \Lambda^{p,m}_{*,w,q}$. From (3.2) in Proposition 3.1 we obtain

$$\frac{\|\nabla_{q,x} f\|_{p,q}}{w(x)} \leq C_q \int_0^\infty \Theta(t,x) \frac{\|\varphi_t *_q f\|_{p,q}}{w(t)} \frac{d_q t}{t}$$

where

$$\Theta(t,x) = \frac{w(t)}{w(x)} \min(1, \frac{x}{t}).$$

Now take

$$(\Omega_1, \mu_1) = (\mathbb{R}_{q,+}, \frac{d_q x}{x})$$

and

$$(\Omega_2, \mu_2) = (\mathbb{R}_{q,+}, \frac{d_q x}{x}).$$

Combine now again Proposition 3.2 and Lemma 2.2 to get the boundedness of T_K from $L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})$ into $L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})$. Therefore,

$$\begin{aligned}
\|f\|_{\Lambda^{p,m}_{*,w,q}} &\leq C_q \|T_K(\frac{\|\varphi_t *_q f\|_{p,q}}{w(t)})\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q x}{x})} \\
&\leq C_q \|\frac{\|\varphi_t *_q f\|_{p,q}}{w(t)}\|_{L^m(\mathbb{R}_{q,+}, \frac{d_q t}{t})} \\
&\leq C_q \|f\|_{B^{p,m}_{*,w,\varphi,q}}.
\end{aligned}$$

□

References

[1] L. D. Abreu, *Functions q-orthogonal with respect to their own zeros*, Proc. Amer. Math. Soc. **134**(2006), 2695-2701.

[2] G. E. Andrews, *q-Series: their development in analysis number theory, combinatorics, physics and computer algebra*, CBMS Series, Amer. Math. Soc. Providence, RI, **66**(1986), 223–241.

[3] J. L. Ansorna and O. Blasco, *Characterization of weighted Besov spaces*, Math. Nachr., **171**(1995), 5–17

[4] O.V. Besov, *On a family of function spaces in connection with embeddings and extentions*, Trudy Math. Inst. Steklov, **60** (1966), 42–81.

[5] M. Bohner, M. Fan and J. Zhang, *Periodicity of scalar dynamic equaton and application to population models*, J. Math. Anal. appl. **330** (2007), 1–9 .

[6] M. Bohner, T. Hudson, *Euler-type boundary value problems in quantum calculus* , International Journal of Applied Mathematics and Statistics, **9**(2007), 19–23.

[7] L. Dhaouadi, J. El Kamel and A. Fitouhi, *Positivity of q-even translation and Inequality in q-Fourier analysis*, JIPAM. J. Inequal. Pure Appl. Math **171**(2006), 1–14.

[8] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of mathematics and its applications 35, Cambridge university press, 1990.

[9] I. Gravagne, J. Davis and R. Marks II, *How deterministic must a real time controller be*, Proceedings of 2005, IEEE/RSI International Conference on intelligent Robots and Systems, Alberta, Aug. 2-6, (2005), 3856–3861.

[10] A. Fitouhi and F. Bouzeffour, *q-cosine Fourier transform and q-heat equation*, Ramanjuan J. in press.

[11] A. Fitouhi, L. Dhaouadi and J. El Kamel, *Inequalities in q-Fourier analysis* , J. Inequal. Pure Appl. Math. **171**(2006), 1–14.

[12] A. Fitouhi, M. Hamza and F. Bouzeffour, *The q-*J_α *Bessel function*, J. Approx. Theory **115**(2002), 114–116.

[13] A. Fitouhi and A. Nemri, *Distribution and convolution product in quantum calculus*, Afr. Diaspora. J. Math, **7**(2008), 39–58.

[14] T.M. Flett, *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. and appl, **39**(1972), 125–158.

[15] T. M. Flett, *Temperatures, Bessel potentials and Lipschitz spaces*, Proc. London Math. Soc, **20**(1970), 749–768.

[16] F.H. Jackson, *On q-definite integrals*, Quart. J. Pure. Appl. Math, **41**(1910), 193–203.

[17] V.G. Kac and P. Cheeung, *Quantum calculus*, Universitext, Springer-Verlag, New York, (2002).

[18] A. Nemri and B. Selmi, *Sobolev type spaces in quantum calculus*, J. Math. Anal. Appl, **359**(2009), 588–601.

[19] A. Nemri and B. Selmi, *On a Calderón's formula in quantum calculus*, Indagationes Mathematicae, **24**(2013), 491–504.

[20] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series, NC,(1976).

[21] S. Sanyal, *Stochastic dynamic equation*, PhD Dissertation, , Missouri University of Science and Technology (2008).

[22] Q. Sheng, M. Fadag, J. Henderson and J. Davis, *An exploiration of combined dynamic derivatives on time scales and their applications*, Nonlinear Analysis: Real World Applications, **7**(2006), 395–413.

[23] M. Taibleson, *On the theory of Lipschitz spaces of distributions on euclidean n-space*, I, II,III.

[24] A. Torchinsky, *Real-variable Methods in Harmonics Analysis*, Academic Press , (1986).

[25] H. Triebel, *Theory of functon spaces*, Monographs in Math., vol. 78, Birkuser, Verlag, Basel, (1983).